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A CONSTRUCTIVE PROOF OF TUCKER'S COMBINATORIAL LEMMA

by

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Abstract

Tucker's combinatorial lemma is concerned with certain labellings of the vertices of a triangulation of the n-ball. It can be used as a basis for the proof of antipodal-point theorems in the same way that Sperner's lemma yields Brouwer's theorem. Here we give a constructive proof, which thereby yields algorithms for antipodal-point problems. Our method is based on an algorithm of Reiser.

Introduction

Let B^n denote the n-ball $\{x \in R^n \mid \|x\| \leq 1\}$ where $\|x\|$ is the ℓ_1-norm $\sum_i |x_i|$, and let S^{n-1} denote its boundary $\{x \in R^n \mid \|x\| = 1\}$. We will call special a centrally symmetric triangulation of B^n that refines the octahedral subdivision. The following result was proved for $n = 2$ in [6]; for the general case, see [2, pages 134-141].

Tucker's Combinatorial Lemma. Let the vertices of a special triangulation T of B^n be assigned labels from $\{\pm 1,\ldots,\pm n\}$. If antipodal vertices of T on S^{n-1} receive complementary labels (labels that sum to zero), then T contains a complementary 1-simplex (a 1-simplex whose vertices have complementary labels).

Tucker stated his lemma in a different form; the nonexistence of such a labelling with no complementary 1-simplex was asserted. (In [2] and an earlier abstract [7] a related positive assertion is given; however we have not been able to find a constructive proof of this lemma.) We will prove below that a complementary 1-simplex exists by devising an algorithm that will find one. The algorithm is based on a method of Reiser for the nonlinear complementarity problem [4].

Not only is Tucker's lemma stated in [6] in terms of nonexistence; his derivation of antipodal-point theorems from it was by contradiction. We briefly indicate below how constructive proofs of two such theorems follow from our algorithm.

The Borsuk-Ulam Theorem. If a continuous function maps S^n into R^n, at least one pair of antipodal points is mapped into a single point.

Proof. Let the function be $f\colon S^n \to R^n$ and define $g\colon B^n \to R^n$ by $g(x) = f(x, 1-\|x\|) - f(-x, \|x\|-1)$; note that $g(-x) = -g(x)$ for $x \in S^{n-1}$.

Now for any special triangulation T of B^n, label vertex v $+i$ $(-i)$ if $|g_i(v)| = \max_j |g_j(v)|$ and $g_i(v)$ is positive (negative) (if $g(v) = 0$, we are done). In case of ties, the least such index is chosen. This labelling satisfies the requirements of the lemma and hence T contains a complementary 1-simplex. Let x^* be any limit point of such complementary 1-simplices for a sequence of special triangulations whose meshes approach zero. A continuity argument implies that $g(x^*) = 0$, and hence $(x^*, 1-\|x^*\|)$ and its antipode are mapped by f into the same point.

<u>The Lusternik-Schnirelmann Theorem</u>. If S^n is covered by $n+1$ closed sets, at least one of them contains a pair of antipodal points.

<u>Proof</u>. Let the sets be $C_1, C_2, \ldots, C_{n+1}$. Label the vertices of any special triangulation T of B^n as follows. If v is such a vertex, let $y = (v, 1-\|v\|)$. Determine the least i such that C_i contains y or $-y$. If C_i contains both y and $-y$, the theorem is proved. Also, if $i = n+1$ then since neither y nor $-y$ lie in C_j, $1 \leq j \leq n$, both must lie in C_{n+1} and we are done. Hence assume that for each vertex v, i is between 1 and n and C_i contains one of $y, -y$. Label v $+i$ or $-i$ accordingly. Again, the labelling satisfies the requirements of the lemma and a complementary 1-simplex exists. Similarly, a limiting argument (some complementary pair of labels occurs infinitely often; take any limit point of the corresponding complementary 1-simplices) proves the claim.

Meyerson and Wright [3] and Barany [1] have also given algorithms for the Borsuk-Ulam theorem. Both use vector-labelling, and their algorithms should be more efficient for practical problems; however no computationally tractable way is yet known to implement their techniques for dealing with the theoretical possibility of degeneracy. We note that in an algorithm it would be preferable to use the n-ball induced by the ℓ_∞-norm, $B_\infty^n = \{x \in R^n \mid |x_i| \leq 1 \text{ all } i\}$; several efficient special triangulations of B_∞^n exist, for example, K_1 and J_1 [5, Chapter III]. Figure 1 below shows B_∞^2 rather than B^2 for this reason.

The Algorithm

First we need some notation. We note by $\operatorname{sgn}(\lambda)$ the sign $(0,+1 \text{ or } -1)$ of any real number λ; similarly, for a vector $x = (x_i) \in R^n$, $\operatorname{sgn}(x)$ is the vector $(\operatorname{sgn}(x_i))$. For any sign vector $s \in R^n$ (i.e., each component s_i is 0, $+1$ or -1), $C(s)$ denotes the closure of $\{x \in R^n \mid \operatorname{sgn}(x) = s\}$, i.e. the set of those x for which x_i is nonnegative, zero or nonpositive according as s_i is $1,0$, or -1, for each i. We call $C(s)$ an orthant - actually it is an orthant of a coordinate subspace. Any special triangulation T induces triangulations of $C(s) \cap B^n$ for each s. Let σ be a simplex of T; then $\operatorname{sgn}(x)$ is the same for each x in the relative interior of σ -- we let $\operatorname{sgn}(\sigma)$ be this sign vector. Clearly, $C(\operatorname{sgn}(\sigma))$ is the smallest orthant containing σ. Since T is centrally symmetric, every simplex σ lying in S^{n-1} has an antipodal simplex $-\sigma = \{-x \mid x \in \sigma\}$.

<u>Definition</u>. For any sign vector s, a simplex $\sigma \in T$ is s-labelled if, whenever s_i is nonzero, $s_i\, i$ is a label of some vertex of σ. If σ is $\operatorname{sgn}(\sigma)$-labelled, we say σ is completely labelled.

Note that the 0-simplex $\{0\}$ is always completely labelled by default since its sign vector is zero. Also, if $\sigma \subseteq S^{n-1}$ is completely labelled, so is its antipodal simplex.

The algorithm proceeds by tracing a path in a graph G whose nodes are completely labelled simplices until it finds a complementary 1-simplex. The graph is given by the following:

Definition. Two completely labelled simplices σ and τ are adjacent in G if they both lie in S^{n-1} and are antipodal, or if one is a face of the other and $\sigma \cap \tau$ is $\mathrm{sgn}(\sigma \cup \tau)$-labelled. The degree of a completely labelled simplex is the number of completely labelled simplices adjacent to it in G.

Figure 1 illustrates completely labelled simplices and adjacency for $n = 2$.

Proposition.

(a) The 0-simplex $\{0\}$ has degree 1;

(b) Each completely labelled simplex containing a complementary 1-simplex has degree 1;

(c) Every other completely labelled simplex has degree 2.

Proof. Let s be the sign vector of the completely labelled simplex σ and suppose s has k nonzero components. Then σ lies in the k-dimensional orthant $C(s)$. In addition, the vertices of σ must contain at least k distinct labels, since σ is s-labelled. Hence σ is a (k-1)- or a k-simplex.

Suppose first that σ is a (k-1)-simplex. If σ does not lie in S^{n-1}, it is a face of precisely two k-simplices in $C(s)$, both completely labelled since σ is. If σ lies in S^{n-1}, it is a face of one completely labelled k-simplex in $C(s)$, and its antipode is completely labelled. In either case, σ is of type (c) and has degree 2.

Suppose now σ is a k-simplex. It then has $k+1$ vertices, with one extra label besides the k it is forced to have by completeness. This other label is either a duplicate of one of the k, the complement of one of the k, or $\pm j$ with $s_j = 0$. In the first case, σ has two faces with all k labels and both are completely labelled; σ is of type (c) and has degree 2. In the second case, σ has just one face with the required k labels;

σ is of type (b) and has degree 1. In the last case, suppose the extra label is $+j$ $(-j)$ and let t be a sign vector agreeing with s except that $t_j = +1$ (-1). Then σ is a face of a unique $(k+1)$-simplex in $C(t)$ and this simplex is completely labelled. In addition, σ has one face with the required k labels; the only exception is when σ is the 0-simplex $\{0\}$. Hence σ is either $\{0\}$ and has degree 1 or is of type (c) with degree 2. The proposition is now proved.

The combinatorial lemma follows directly from the proposition, since every graph has an even number of nodes of odd degree. Indeed, we have a stronger result; there is an odd number of completely labelled simplices containing a complementary 1-simplex. However, since some complementary 1-simplices are contained in no completely labelled simplex and others in several, we can say nothing of the parity of complementary 1-simplices.

More than just a proof, we now have an algorithm: *follow a path of adjacent completely labelled simplices from the 0-simplex* $\{0\}$. By the proposition, the path can terminate only when it encounters a complementary 1-simplex. In Figure 1 the sequence is $\rho_0, \tau_1, \sigma_2, \ldots, \tau_{14}$. The algorithm can be stated more concisely as follows; this description corresponds to Reiser's algorithm in all respects except the reflection step.

Algorithm

Step 0 (Initialization). Set $s = v = 0 \in R^n$, $\sigma = \{v\}$. Go to Step 1.

Step 1 (Labelling). Find the label εi of v, $\varepsilon = \pm 1$, $1 \leq i \leq n$. If it is the complement of the label of another vertex of σ, stop. If $s_i = 0$ go to Step 2, otherwise to Step 3.

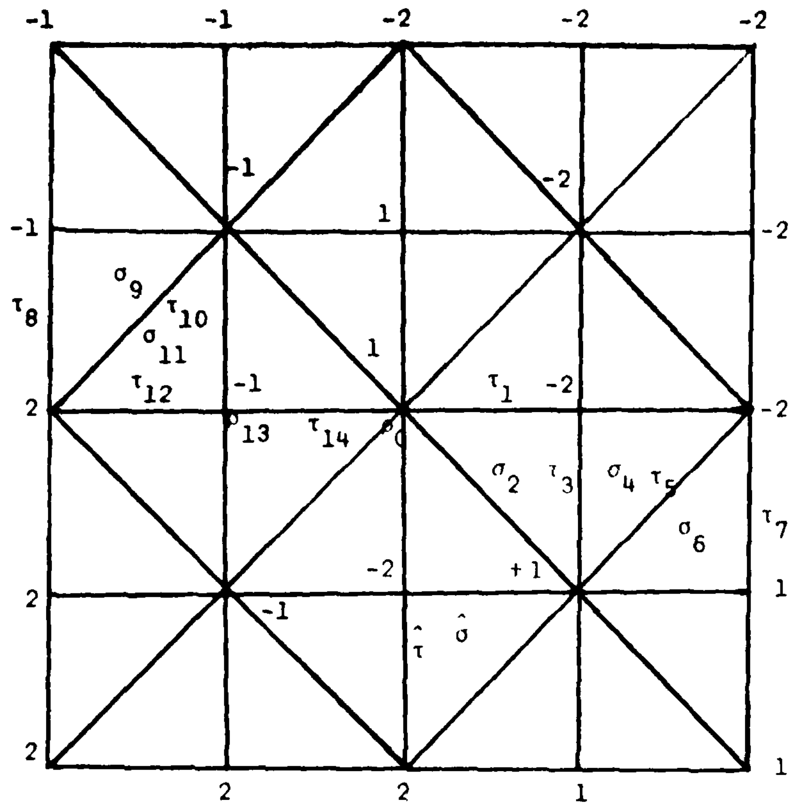
Step 2 (Increasing the dimension). Set $s_i = \varepsilon$. Let τ be the simplex of $C(s)$ with σ as a face, and v its new vertex. Set $\sigma \leftarrow \tau$ and go to Step 1.

Step 3 (Dropping a vertex). Find the vertex $w \neq v$ of σ with label εi. Let τ be the face of σ opposite w. If τ lies in S^{n-1}, set $\sigma \leftarrow -\sigma$, $s \leftarrow -s$, $v \leftarrow -w$ and go to Step 1. If $\mathrm{sgn}(\tau) = s$, let ρ be the simplex in $C(s)$ with τ as a face distinct from σ, and let v be its new vertex; set $\sigma \leftarrow \rho$ and go to Step 1. Otherwise go to Step 4.

Step 4 (Decreasing the dimension). Find i with $(\mathrm{sgn}(\tau))_i = 0$ and $s_i \neq 0$. Set $\sigma \leftarrow \tau$, $\varepsilon \leftarrow s_i$, $s \leftarrow \mathrm{sgn}(\tau)$, $v \leftarrow w$ and go to Step 3.

In this form the algorithm generates $\rho_0, \tau_1, \sigma_2, \sigma_4, \sigma_6, \sigma_9, \sigma_{11}, \tau_{12}$ and τ_{14} as its successive σ's in Figure 1.

Note that the only requirement on the labelling is that antipodal vertices in S^{n-1} have complementary labels--the coordinate structure of B^n is immaterial. Hence the labels can be permuted so that complementary labels remain complementary and the algorithm run again. This is a possible method for obtaining several complementary 1-simplices--but of course it is not guaranteed to find more than one. In Figure 1, if we interchange labels 1 and -2, -1 and 2 then the completely labelled simplex $\hat{\sigma}$ containing the complementary 1-simplex $\hat{\tau}$ is generated.



All simplices marked are completely *labelled*. σ's *are* 2-simplices, τ's 1-simplices and ρ's 0-simplices. τ_{14} and $\hat{\tau}$ are complementary. All simplices with consecutive indices are adjacent. Since $\hat{\tau}$ is not (1,-1)-labelled, $\hat{\sigma}$ and $\hat{\tau}$ are not adjacent.

Figure 1

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